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***Abstract*—The Bertrand Oligopoly situation with Shubik's demand functions is modelled as a cooperative TU game. For that purpose two optimization problems are solved to arrive at the description of the worth of any coalition in the so-called Bertrand Oligopoly Game. Under certain circumstances, this Bertrand oligopoly game has clear affinities with the well-known notion in statistics called variance with respect to the distinct marginal costs. This Bertrand Oligopoly Game is shown to be totally balanced, but fails to be convex unless all the firms have the same marginal costs. Under the complementary circumstances, the Bertrand Oligopoly Game is shown to be convex and in addition, its Shapley value is fully determined on the basis of linearity applied to an appealing decomposition of the Bertrand Oligopoly Game into the difference between two convex games, besides two non-essential games. One of these two essential games concerns the square of one non-essential game.**

***Index Terms*—Bertrand Oligopoly situation, Bertrand Oligopoly Game, Convexity, Shapley Value, Total Balancedness.**

I. INTRODUCTION

A central problem in oligopoly theory is the existence of collusive behaviors between firms, that is, situations in which firms are able to coordinate and to stabilize their strategies in order to increase their profits. The classical Cournot and Bertrand oligopoly situations are such examples where firms are better off through cooperation rather than by acting independently. A cartel operating successfully is the OPEC (Organization of the Petroleum Exporting Countries) cartel which restricts oil supply in order to control oil price market. Another example of a cartel which had operated is the agreement between multinational firms Saint-Gobain, Pilkington, Asahi and Soliver in flat glass industry. Their illegal agreement on the price of glass in the car industry from 1998 to 2003 had been fined by the European Commission in 2008. Non-cooperative game theory has provided the theoretical bases for the existence of collusive behaviors between firms by means of repeated games. Under this approach, each firm does not have any interest in defecting from the collusive behavior because it rationally anticipates future punishments in the periods following its defection. We refer to Pepall et al. (2008) for an overview on this topic. An alternative way to formalize the existence of collusive behaviors comes from cooperative game theory. Under this approach, firms are allowed to sign binding agreements in order to form cartels called coalitions. With such an assumption cooperative games called oligopoly TU(Transferable Utility)-games can be defined and the existence of collusive behaviors is then related to the non-emptiness of the core of such games. Aumann(1959) proposes two approaches in order to define cooperative game: according to the first, every cartel computes the total profit which it can guarantee itself regardless of what outsiders do; the second approach consists in computing the minimal profit for which outsiders can prevent the firms in the cartel from getting more. These two assumptions lead to consider the α and β-characteristic functions respectively. In this article, we follow this cooperative approach to analyze collusive behaviors and we study a subclass of oligopoly TU-games in α and β-characteristic function forms. Many works have studied the core of oligopoly TU-games. As regards Cournot oligopoly TU-games with or without transferable technologies, Zhao (1999a,b) shows that the α and β-characteristic functions lead to the same class of Cournot oligopoly TU-games. When technologies are transferable, Zhao(1999a) provides a necessary and sufficient condition to establish the convexity property in

case the inverse demand function and cost functions are linear. Although these games may fail to be convex in general, Norde et al. (2002) show they are nevertheless totally balanced. When technologies are not transferable, Zhao(1999b) proves that the core of such games is non-empty if every individual profit function is continuous and concave. Furthermore, Norde et al.(2002) show that these games are convex in case the inverse demand function and cost functions are linear, and Driessen and Meinhardt(2005) provide economically meaningful sufficient conditions to guarantee the convexity property in a more general case.
As regards Bertrand oligopoly TU-games, Deneckere and Davison (1985) consider a Bertrand oligopoly situation with differentiated products in which the demand system is Shubik's (1980) and firms operate at a constant and identical marginal and average cost. They prove that these games have a superadditive property in the sense that a merger of two disjoint cartels results in a joint after-merger profit for them which is greater than the sum of their pre-merger profits. Lardon(2010) extends this result by considering the α and β-characteristic functions of these games. As for Cournot oligopoly TU-games, he shows that the α and β-characteristic functions lead to the same class of Bertrand oligopoly TU-games and proves that the convexity property holds for this class of games.

In this article, we study Bertrand oligopoly TU-games in α and β-characteristic function forms with Shubik's demand functions in which firms have possibly distinct marginal costs. We prove that the α and β-characteristic functions lead to the same class of Bertrand oligopoly TU-games, and so we extend Lardon 's result (2010) which holds only with identical marginal costs of firms. For this class of games, we show that if the intercept of demand is sufficiently small, then Bertrand oligopoly TU-games share clear similarities with a well-known notion in statistics called variance with respect to the distinct marginal costs. Such games are shown to be totally balanced, but fail to be convex unless all the firms have the same marginal costs. Moreover, we prove that if the intercept of demand is sufficiently large, then Bertrand oligopoly TU-games are convex and in addition, the Shapley value is fully determined on the basis of linearity applied to an appealing decomposition of Bertrand oligopoly TU-games into the difference between two convex games, besides two non-essential games. One of these two convex games is defined as the square of one of the two non-essential games.
This article is structured as follows. In section 2, we introduce the model and show that Bertrand oligopoly TU-games in β-characteristic function form with distinct marginal costs are well-defined. Moreover, we prove that the α and β-characteristic functions lead to the same class of Bertrand oligopoly TU-games. In section 3, we show that if the intercept of demand is sufficiently large, then Bertrand oligopoly TU-games are convex. In section 4, we study and characterize the Shapley value on the class of Bertrand oligopoly TU-games. In section 5, we show that if the intercept of demand is sufficiently small, then Bertrand oligopoly TU-games fail to be convex but are nevertheless totally balanced. Section 6 gives some concluding remarks.

II. The non-symmetric Bertrand Oligopoly TU-Game with Shubik's Demand Functions

A *Bertrand oligopoly situation* is described by a 3-tuple $\langle N, (D_i)_{i\in N}, (C_i)_{i\in N}\rangle$, where $N = \{1, 2, \ldots, n\}$ is the finite *set of firms*, such that, for every firm $i \in N$, the *Shubik's demand function* $D_i : \mathbb{R}^n_+ \to \mathbb{R}$ and the *(linear) cost function* $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ with marginal cost $c_i \geq 0$ respectively are given by ([9])

$$D_i(p_1, p_2, \ldots, p_n) = V - p_i - r \cdot \left[p_i - \tfrac{1}{n} \cdot \sum_{k\in N} p_k\right] \cdot x \quad (1)$$

and $C_i(x) = c_i$ for all $x \in \mathbb{R}_+$ where p_i is the *price* charged by firm i, the *demand's intercept* $V \geq 0$ when all prices are zero, and let $r > 0$ be the *substitutability parameter*. When r approaches zero, products become unrelated, and when r approaches infinity, products become perfect substitutes. The quantity demanded of firm i's brand depends on its own price p_i and the difference between its own price and the average price in the industry. The latter quantity is decreasing with respect to its own price p_i and increasing with respect to any price p_j, $j \neq i$. Notice that firms may operate at possibly different marginal costs $c_i \geq 0$, $i \in N$, and these marginal costs do not limit the non-negative prices $p_i \geq 0$, $i \in N$, of firms.
The corresponding *Bertrand Oligopoly Game in normal form* $\langle N, (X_i)_{i\in N}, (\pi_i)_{i\in N}\rangle$ is given by player i's *strategy set* $X_i = \mathbb{R}_+ = [0, \infty)$ and *individual profit function* $\pi_i : \Pi_{k\in N} X_k \to \mathbb{R}$ such that

$$\pi_i(p_1, p_2, \ldots, p_n) = (p_i - c_i) \cdot D_i(p_1, p_2, \ldots, p_n)$$

So, for all $i \in N$,

$$\begin{aligned} & \pi_i(p_1, p_2, \ldots, p_n) \\ = \quad & (p_i - c_i) \cdot \left[V - (1 + r) \cdot p_i + \tfrac{r}{n} \cdot \sum_{k\in N} p_k\right] \quad (2) \end{aligned}$$

Denote for any $T \subseteq N$, $T \neq \emptyset$, the coalitional strategy set $X_T = \Pi_{k\in T} X_k$ and define the *coalitional profit* *function* $\pi_T : X_T \times X_{N\setminus T} \to \mathbb{R}$ by $\pi_T(p_T, p_{N\setminus T}) = \sum_{k\in T} \pi_k(p_T, p_{N\setminus T})$, for all $(p_T, p_{N\setminus T}) \in X_T \times X_{N\setminus T}$. The corresponding *Bertrand Oligopoly Game in α- and β-characteristic function form* $\langle N, v_\alpha\rangle$ and $\langle N, v_\beta\rangle$ are defined, for every coalition $S \subseteq N$, $S \neq \emptyset$, as follows:

$$v_\alpha(S) = \max_{p_S\in X_S} \min_{p_{N\setminus S}\in X_{N\setminus S}} \pi_S(p_S, p_{N\setminus S}) \tag{3}$$

$$v_\beta(S) = \min_{p_{N\setminus S}\in X_{N\setminus S}} \max_{p_S\in X_S} \pi_S(p_S, p_{N\setminus S}) \tag{4}$$

$$\pi_S(p_S, p_{N\setminus S}) = \sum_{j\in S} \pi_j(p_S, p_{N\setminus S})$$
$$= \sum_{j\in S}(p_j - c_j)\cdot\Big[V - (1+r)\cdot p_j + \tfrac{r}{n}\cdot\sum_{k\in N} p_k\Big] \tag{5}$$

Generally speaking, it always holds $v_\alpha(S) \leq v_\beta(S)$ as well as $v_\alpha(N) = v_\beta(N)$. In the remainder we pay attention to the Bertrand oligopoly game in β-characteristic function form and let $\bar{c}_S = \frac{1}{s}\cdot\sum_{j\in S} c_j$ denote the *average coalitional cost* of coalition $S \subseteq N$, $S \neq \emptyset$.

Theorem 2.1: Fix coalition $S \subseteq N$, $S \neq \emptyset$, and strategy profile $p_{N\setminus S} \in X_{N\setminus S}$.

(i) Concerning the maximization program $\max_{p_S\in X_S} \pi_S(p_S, p_{N\setminus S})$, the first order conditions are given by $\frac{\partial\pi_S}{\partial p_i}(p_S, p_{N\setminus S}) = 0$ for all $i \in S$ and its unique solution $\bar{p}_i^S$ is given by

$$\bar{p}_i^S = \bar{p}_S + \tfrac{c_i}{2} \text{for all } i \in S\text{, where} \tag{6}$$

$$\bar{p}_S = \tfrac{1}{2}\cdot\left[\frac{n\cdot V + r\cdot\sum_{k\in N\setminus S} p_k}{n + r\cdot(n-s)}\right] \tag{7}$$

(ii) Concerning the quadratic minimization program $\min_{p_{N\setminus S}\in X_{N\setminus S}} \pi_S((\bar{p}_j^S)_{j\in S}, p_{N\setminus S})$, where $S \neq N$, its solution is given by

$$\sum_{k\in N\setminus S} \bar{p}_k = \max\Big[0,\quad \tfrac{1}{r}\cdot\Big[\Big[n + r\cdot(n-s)\Big]\cdot\bar{c}_S - n\cdot V\Big]\Big] \tag{8}$$

We distinguish two cases.

(iii) If $n\cdot V \leq [n + r\cdot(n-s)]\cdot\bar{c}_S$, then the following holds: for all $i \in S$,

$$\bar{p}_i^S = \frac{\bar{c}_S + c_i}{2} \quad\text{and}\quad \bar{p}_S = \frac{\bar{c}_S}{2} \tag{9}$$

$$\sum_{k\in N\setminus S} \bar{p}_k = \tfrac{1}{r}\cdot\Big[\Big[n + r\cdot(n-s)\Big]\cdot\bar{c}_S - n\cdot V\Big] \tag{10}$$

$$v_\beta(S) = \tfrac{1+r}{4}\cdot\Big[\sum_{j\in S}(c_j)^2 - s\cdot(\bar{c}_S)^2\Big]$$
$$= \tfrac{1+r}{4}\cdot\sum_{j\in S}\Big[c_j - \bar{c}_S\Big]^2 \tag{11}$$

(iv) If $n\cdot V > [n + r\cdot(n-s)]\cdot\bar{c}_S$, then the following holds: for all $i \in S$

$$\bar{p}_i^S = \tfrac{1}{2}\cdot\Big[\tfrac{n\cdot V}{n + r\cdot(n-s)} + c_i\Big] > \tfrac{\bar{c}_S + c_i}{2} \tag{12}$$

$$\sum_{k\in N\setminus S} \bar{p}_k = 0 \tag{13}$$

$$v_\beta(S) = \tfrac{s}{4\cdot n}\cdot\frac{(n\cdot V)^2}{\Big[n + r\cdot(n-s)\Big]} - \tfrac{V}{2}\cdot\sum_{j\in S} c_j +$$
$$+ \tfrac{1+r}{4}\cdot\sum_{j\in S}(c_j)^2 - \tfrac{r}{4\cdot n}\cdot\Big[\sum_{j\in S} c_j\Big]^2 \tag{14}$$
$$= \tfrac{s}{4\cdot n}\cdot\frac{\Big[n\cdot V - \Big[n + r\cdot(n-s)\Big]\cdot\bar{c}_S\Big]^2}{\Big[n + r\cdot(n-s)\Big]} +$$
$$+ \tfrac{1+r}{4}\cdot\Big[\sum_{j\in S}(c_j)^2 - s\cdot(\bar{c}_S)^2\Big] \tag{15}$$

Remark 2.1: With every collection $\mathcal{C} = (c_i)_{i\in N}$ of marginal costs, there is associated the so-called *Variance* TU game $\langle N, VAR_{\mathcal{C}}\rangle$ of which the characteristic function $VAR_{\mathcal{C}} : \mathcal{P}(N) \to \mathbb{R}$ is defined by $VAR_{\mathcal{C}}(\emptyset) = 0$ and

$$VAR_{\mathcal{C}}(S) = \sum_{j\in S}\Big[c_j - \bar{c}_S\Big]^2 \qquad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{16}$$

The notion of variance is well-known in the field of statistics and it refers to the deviations by individuals from the average coalitional cost measured as the sum of squares of member's deviations. According to (11), the Bertrand Oligopoly Game resembles the associated Variance TU game. For instance, if the marginal costs are given by $c_1 = c_2 = c$ and $c_3 = c + 1$. Then the zero-normalized three-firm Variance Game is given by $v(S) = 0$ if $3 \notin S$ and $v(S) = \frac{s-1}{s}$ if $3 \in S$. That is, $v(\{1,2\}) = 0$, $v(\{1,3\}) = v(\{2,3\}) = \frac{1}{2}$ and

$v(N) = \frac{2}{3}$. Note that firms 1 and 2 are substitutes in this variance game.
Generally speaking, if $i \in N$ and $S \subseteq N$, $S \ni i$, satisfy $c_i = \bar{c}_S$, then it follows immediately $c_i = \bar{c}_{S\backslash\{i\}}$ and in turn, $VAR_{\mathcal{C}}(S\backslash\{i\}) = VAR_{\mathcal{C}}(S)$. Particularly, if $i \in N$ satisfies $c_i = \bar{c}_N$, then it holds $VAR_{\mathcal{C}}(N) - VAR_{\mathcal{C}}(N\backslash\{i\}) = 0$. Because the latter expression represents an upper bound for the core of the zero-normalized Variance game, any core allocation to player i degenerates to zero, provided the marginal cost of firm i coincides with the average grand coalitional cost. Nevertheless, in Section V we prove the balancedness for Variance TU games (or equivalently, the non-emptiness of the core).

Corollary 2.1: Consider the symmetric Bertrand Oligopoly Game $\langle N, v_\beta\rangle$ in that $c_i = c > 0$ for all $i \in N$. Fix coalition $S \subseteq N$, $S \neq \emptyset$.

(i) If $n \cdot V \leq [n + r \cdot (n-s)] \cdot c$, then $v_\beta(S) = 0$ due to $\bar{p}_i^S = c$ for all $i \in S$.

(ii) Define the *proportionally aggregate netto demand* E through $r \cdot c \cdot E = n \cdot (V - c)$. If $n \cdot V > [n + r \cdot (n-s)] \cdot c$, then

$$v_\beta(S) = \tfrac{s}{4 \cdot n} \cdot \frac{\left[n \cdot V - \left[n + r \cdot (n-s)\right] \cdot c\right]^2}{\left[n + r \cdot (n-s)\right]} > 0 \tag{17}$$

or equivalently, if $E > n - s$, then

$$v_\beta(S) = \tfrac{s}{4 \cdot n} \cdot (r \cdot c)^2 \cdot \frac{\left[E - (n-s)\right]^2}{\left[n + r \cdot (n-s)\right]} > 0 \tag{18}$$

The non-zero coalitional worth in the symmetric Bertrand Oligopoly Game depends on the validity of the constraint $n \cdot V > [n + r \cdot (n-s)] \cdot c$ involving the intercept V or the equivalent constraint $E > n - s$ involving the proportionally aggregate netto demand E. In this setting, we interpret $n \cdot (V - c)$ as the aggregate netto demand when prices are zero. Obviously, if a coalition S of size s meets the constraint $E \leq n - s$ yielding zero worth $v_\beta(S) = 0$, then any coalition of the same size s or less inherits the same constraint yielding zero worth. Similarly, if a coalition T of size t meets the inverse constraint $E > n - t$ yielding non-zero worth $v_\beta(T) > 0$, then any coalition of the same size or more inherits the same inverse constraint yielding non-zero worth. According to (19), the per-capita worth $\frac{v_\beta(S)}{s}$ is strategically equivalent to the quotient of the square of a bankruptcy game (with estate E and unitary claims) and a linearly decreasing symmetric game (varying from levels $(1 + r) \cdot n$ down to level n).

The non-symmetric Bertrand Oligopoly Game consists of two types of coalitions because the validity of the relevant constraint $n \cdot V > [n + r \cdot (n-s)] \cdot \bar{c}_S$ involves the average coalitional cost $\bar{c}_S$. According to (11), if the average coalitional cost is sufficiently large, then the coalitional worth is fully determined by the multiple $\frac{1+r}{4}$ of the *Variance* TU Game induced by the distinct marginal costs. Otherwise, according to (14), if the average coalitional cost is sufficiently small, then the coalitional worth in the non-symmetric Bertrand Oligopoly Game counts, besides the associated variance, the non-zero worth in the symmetric Bertrand Oligopoly Game, with the understanding that the constant marginal cost is to be replaced by the average coalitional cost. The rather appealing alternative decomposition (14) into four types of games will be exploited in the Sections IV and V.

Proof of Theorem 2.1

Part 1. Let $i \in S$. The partial derivative of the coalitional profit function (5) is as follows:

$$\begin{aligned}
&\frac{\partial \pi_S}{\partial p_i}(p_S, p_{N\backslash S}) \\
=\;& \left[V - (1+r) \cdot p_i + \tfrac{r}{n} \cdot \sum_{k \in N} p_k\right] + \\
+\;& (p_i - c_i) \cdot \left[-(1+r) + \tfrac{r}{n}\right] + \sum_{j \in S\backslash\{i\}} (p_j - c_j) \cdot \tfrac{r}{n} \\
=\;& V - (1+r) \cdot (2 \cdot p_i - c_i) + \\
+\;& \tfrac{r}{n} \cdot \sum_{j \in S} (p_j - c_j) + \tfrac{r}{n} \cdot \sum_{k \in N} p_k
\end{aligned}$$

Consequently, the solution to the first order condition $\frac{\partial \pi_S}{\partial p_i} = 0$ satisfies, for all $i \in S$,

$$p_i = \tfrac{c_i}{2} + \tfrac{1}{2 \cdot (1+r)} \cdot \left[V + \tfrac{r}{n} \cdot \left[\sum_{j \in S} (p_j - c_j) + \sum_{k \in N} p_k\right]\right].$$

So far, we conclude that for the solution to the first order conditions (associated with the maximization program) it holds that $p_i - \frac{c_i}{2}$ is constant for all $i \in S$, say $p_i - \frac{c_i}{2} = \bar{p}_S$ for every $i \in S$. Through substitution in the latter expression, we arrive at the following relationships:

$$\begin{aligned}
& 2 \cdot (1+r) \cdot \bar{p}_S \\
=\;& V + \tfrac{r}{n} \cdot \left[\sum_{k \in N\backslash S} p_k + \sum_{j \in S} (2 \cdot p_j - c_j)\right] \\
=\;& V + \tfrac{r}{n} \cdot \left[\sum_{k \in N\backslash S} p_k + 2 \cdot s \cdot \bar{p}_S\right]
\end{aligned}$$

Rewriting the latter equality yields

$$\left[2\cdot(1+r)-\tfrac{2\cdot r\cdot s}{n}\right]\cdot\bar{p}_S=V+\tfrac{r}{n}\cdot\sum_{k\in N\setminus S}p_k$$

and thus, $\bar{p}_S=\frac{1}{2}\cdot\left[\frac{n\cdot V+r\cdot\sum_{k\in N\setminus S}p_k}{n+r\cdot(n-s)}\right]$ Recall that the solution $\bar{p}_i^S$ to the first order conditions is given by $\bar{p}_i^S=\bar{p}_S+\frac{c_i}{2}$ for all $i\in S$.

Part 2. First method.

For the sake of the minimization program $\min_{p_{N\setminus S}\in X_{N\setminus S}}\pi_S((\bar{p}_j^S)_{j\in S},p_{N\setminus S})$, we derive from (5) that its objective function reduces as follows:

$$\begin{aligned}&\pi_S((\bar{p}_j^S)_{j\in S},p_{N\setminus S})\\=&\sum_{j\in S}(\bar{p}_j^S-c_j)\cdot\Bigg[V-(1+r)\cdot\bar{p}_j^S+\\+&\tfrac{r}{n}\cdot\sum_{k\in S}\bar{p}_k^S+\tfrac{r}{n}\cdot\sum_{k\in N\setminus S}p_k\Bigg]\end{aligned}\tag{19}$$

Note that, by (6), each partial derivative of $\bar{p}_i^S$, $i\in S$, is the same, given by

$$\frac{\partial\bar{p}_i^S}{\partial p_\ell}=\frac{r}{2\cdot\Big[n+r\cdot(n-s)\Big]}=\delta_s\quad\text{for all }\ell\in N\setminus S.$$

Hence, by differentiating (19),

$$\begin{aligned}&\frac{\partial\pi_S}{\partial p_\ell}((\bar{p}_j^S)_{j\in S},p_{N\setminus S})\\=&\sum_{j\in S}\delta_s\cdot\Bigg[V-(1+r)\cdot\bar{p}_j^S+\tfrac{r}{n}\cdot\sum_{k\in S}\bar{p}_k^S\\+&\tfrac{r}{n}\cdot\sum_{k\in N\setminus S}p_k\Bigg]+\sum_{j\in S}(\bar{p}_j^S-c_j)\cdot\Bigg[-(1+r)\cdot\delta_s\\+&\tfrac{r}{n}\cdot s\cdot\delta_s+\tfrac{r}{n}\Bigg]\\=&\delta_s\cdot\Bigg[s\cdot V+\Big[\tfrac{r\cdot s}{n}-(1+r)\Big]\cdot\sum_{k\in S}\bar{p}_k^S\\+&\tfrac{r\cdot s}{n}\cdot\sum_{k\in N\setminus S}p_k\Bigg]\\+&\Bigg[\sum_{j\in S}\bar{p}_j^S-s\cdot\bar{c}_S\Bigg]\cdot\Bigg[\delta_s\cdot\Big[\tfrac{r\cdot s}{n}-(1+r)\Big]+\tfrac{r}{n}\Bigg]\\=&\delta_s\cdot\Bigg[s\cdot V+\tfrac{r\cdot s}{n}\cdot\sum_{k\in N\setminus S}p_k\Bigg]-\tfrac{r}{2\cdot n}\sum_{k\in S}\bar{p}_k^S\\+&\Bigg[\sum_{j\in S}\bar{p}_j^S-s\cdot\bar{c}_S\Bigg]\cdot\Bigg[\frac{-r}{2\cdot n}+\frac{r}{n}\Bigg]\\=&\delta_s\cdot\Bigg[s\cdot V+\tfrac{r\cdot s}{n}\cdot\sum_{k\in N\setminus S}p_k\Bigg]-\tfrac{r\cdot s}{2\cdot n}\cdot\bar{c}_S\end{aligned}$$

Concerning the solution to the minimization problem, it follows from the first order conditions that for all $\ell\in N\setminus S$

$$\begin{aligned}&\frac{\partial\pi_S}{\partial p_\ell}((\bar{p}_j^S)_{j\in S},p_{N\setminus S})=0\\\text{iff}\quad&\delta_s\cdot\Bigg[V+\tfrac{r}{n}\cdot\sum_{k\in N\setminus S}p_k\Bigg]-\tfrac{r}{2\cdot n}\cdot\bar{c}_S=0\\\text{iff}\quad&r\cdot\sum_{k\in N\setminus S}p_k=\tfrac{r}{2}\cdot\tfrac{\bar{c}_S}{\delta_s}-n\cdot V\end{aligned}$$

Finally take care about the non-negativity constraint for prices. This completes the proof of (8) according to the first method.

Part 2. Second method.

For the sake of the minimization program $\min_{p_{N\setminus S}\in X_{N\setminus S}}\pi_S((\bar{p}_j^S)_{j\in S},p_{N\setminus S})$, we derive from (5) that its objective function partially reduces as follows: for all $j\in S$

$$\begin{aligned}&V-(1+r)\cdot\bar{p}_j^S+\tfrac{r}{n}\cdot\sum_{k\in N}\bar{p}_k\\=&V-(1+r)\cdot\Big[\bar{p}_S+\tfrac{c_j}{2}\Big]\\+&\tfrac{r}{n}\cdot\sum_{k\in S}\Big[\bar{p}_S+\tfrac{c_k}{2}\Big]+\tfrac{r}{n}\cdot\sum_{k\in N\setminus S}p_k\\=&V-\Big[1+r-\tfrac{s\cdot r}{n}\Big]\cdot\bar{p}_S-(1+r)\cdot\tfrac{c_j}{2}\\+&\tfrac{r\cdot s}{2\cdot n}\cdot\bar{c}_S+\tfrac{r}{n}\cdot\sum_{k\in N\setminus S}p_k\\=&-\tfrac{1}{n}\cdot\Big[n+r\cdot(n-s)\Big]\cdot\bar{p}_S-(1+r)\cdot\tfrac{c_j}{2}\\+&\tfrac{r\cdot s}{2\cdot n}\cdot\bar{c}_S+\tfrac{2}{n}\cdot\Big[n+r\cdot(n-s)\Big]\cdot\bar{p}_S\\=&\tfrac{1}{n}\cdot\Big[n+r\cdot(n-s)\Big]\cdot\bar{p}_S-(1+r)\cdot\tfrac{c_j}{2}+\tfrac{r\cdot s}{2\cdot n}\cdot\bar{c}_S\end{aligned}$$

by using (6). Thus, by treating $\bar{p}_S$ as the variable, the objective function is described by

$$\begin{aligned}
&\pi_S((\bar{p}_j^S)_{j\in S}, p_{N\setminus S})\\
=\;& \sum_{j\in S}(\bar{p}_j^S - c_j)\cdot\Bigg[V-(1+r)\cdot\bar{p}_j^S+\tfrac{r}{n}\cdot\sum_{k\in N}\bar{p}_k\Bigg]\\
=\;& \sum_{j\in S}(\bar{p}_S - \tfrac{c_j}{2})\cdot\Bigg[\tfrac{1}{n}\cdot\Bigg[n+r\cdot(n-s)\Bigg]\cdot\bar{p}_S\\
-\;& (1+r)\cdot\tfrac{c_j}{2}+\tfrac{r\cdot s}{2\cdot n}\cdot\bar{c}_S\Bigg]
\end{aligned}$$

Obviously, by the latter equality, the objective function $\pi_S((\bar{p}_j)_{j\in S}, p_{N\setminus S})$ is a quadratic function of $\bar{p}_S$, say of the form $\alpha_2\cdot(\bar{p}_S)^2+\alpha_1\cdot\bar{p}_S+\alpha_0$. Clearly, the coefficient of the quadratic term $(\bar{p}_S)^2$ equals $\alpha_2=\frac{s}{n}\cdot\Bigg[n+r\cdot(n-s)\Bigg]>0$, while the coefficient of the term $\bar{p}_S$ equals

$$\begin{aligned}
\alpha_1 &= -\sum_{j\in S}c_j\cdot\tfrac{1}{2\cdot n}\cdot\Bigg[n+r\cdot(n-s)\Bigg]\\
&- \tfrac{1+r}{2}\cdot\sum_{j\in S}c_j+\tfrac{r\cdot s^2}{2\cdot n}\cdot\bar{c}_S\\
&= \frac{-\bar{c}_S}{2\cdot n}\cdot\Bigg[s\cdot\Bigg[n+r\cdot(n-s)\Bigg]\\
&+ (1+r)\cdot s\cdot n - r\cdot s^2\Bigg]\\
&= \tfrac{-s}{n}\cdot\Bigg[n+r\cdot(n-s)\Bigg]\cdot\bar{c}_S
\end{aligned}$$

and the constant term $\alpha_0=\frac{1+r}{4}\cdot\sum_{j\in S}(c_j)^2-\frac{r\cdot s^2}{4\cdot n}\cdot(\bar{c}_S)^2$. This quadratic function attains its minimum for $\bar{p}_S=\frac{-\alpha_1}{2\cdot\alpha_2}=\frac{\bar{c}_S}{2}$ or equivalently, $\sum_{k\in N\setminus S}p_k=\frac{1}{r}\cdot\Bigg[\Bigg[n+r\cdot(n-s)\Bigg]\cdot\bar{c}_S-n\cdot V\Bigg]$. Finally take care about the non-negative constraint for prices. This completes the proof of (8) according to the second method.

Part 3. First assumption. Suppose $n\cdot V\le[n+r\cdot(n-s)]\cdot\bar{c}_S$. Under these circumstances, we derive from (8) and in turn, through substitution in (6),

$$r\cdot\sum_{k\in N\setminus S}\bar{p}_k = \Bigg[n+r\cdot(n-s)\Bigg]\cdot\bar{c}_S-n\cdot V$$

as well as$\bar{p}_S=\frac{\bar{c}_S}{2}$ and so,

$$\bar{p}_i^S=\bar{p}_S+\frac{c_i}{2}=\frac{\bar{c}_S+c_i}{2}$$

as well as $\bar{p}_i^S-c_i=\frac{\bar{c}_S-c_i}{2}$ for all $i\in S$.
Further, for all $j\in S$, the objective function partially reduces as follows:

$$\begin{aligned}
&V-(1+r)\cdot\bar{p}_j^S+\tfrac{r}{n}\cdot\sum_{k\in N}\bar{p}_k\\
=\;& V-(1+r)\cdot\frac{\bar{c}_S+c_j}{2}+\tfrac{r}{n}\cdot\sum_{j\in S}\tfrac{\bar{c}_S+c_j}{2}\\
+\;& \tfrac{1}{n}\cdot\Bigg[\Bigg[n+r\cdot(n-s)\Bigg]\cdot\bar{c}_S-n\cdot V\Bigg]\\
=\;& -(1+r)\cdot\frac{\bar{c}_S+c_j}{2}+\tfrac{r\cdot s}{n}\cdot\bar{c}_S\\
+\;& \tfrac{1}{n}\cdot\Bigg[n+r\cdot(n-s)\Bigg]\cdot\bar{c}_S
\end{aligned}$$

Through substitution in (5), the coalitional profit reduces as follows:

$$\begin{aligned}
v_\beta(S) &= \pi_S((\bar{p}_j^S)_{j\in S}, \bar{p}_{N\setminus S})\\
&= \sum_{j\in S}\frac{\bar{c}_S-c_j}{2}\cdot\Bigg[-(1+r)\cdot\frac{\bar{c}_S+c_j}{2}+\tfrac{r\cdot s}{n}\cdot\bar{c}_S\\
&+ \tfrac{1}{n}\cdot\Bigg[n+r\cdot(n-s)\Bigg]\cdot\bar{c}_S\Bigg]\\
&= -\tfrac{1+r}{4}\cdot\sum_{j\in S}(\bar{c}_S-c_j)\cdot c_j\\
&= \tfrac{1+r}{4}\cdot\Bigg[\sum_{j\in S}(c_j)^2-s\cdot(\bar{c}_S)^2\Bigg]
\end{aligned}$$

This proves (11) concerning the coalitional worth $v_\beta(S)$.

Part 3. Second assumption. Suppose $n\cdot V>[n+r\cdot(n-s)]\cdot\bar{c}_S$. Under these circumstances, we derive from (8) that $\sum_{k\in N\setminus S}\bar{p}_k=0$, and in turn, through substitution in (6),

$$\bar{p}_S = \tfrac{1}{2}\cdot\tfrac{n\cdot V}{[n+r\cdot(n-s)]}$$

Recall that $\bar{p}_i^S=\bar{p}_S+\frac{c_i}{2}$ for all $i\in S$. Further, for all

$j \in S$, the objective function partially reduces as follows:

$$\begin{aligned}
& V-(1+r)\cdot\bar{p}_j^S+\tfrac{r}{n}\cdot\sum_{k\in N}\bar{p}_k \\
= \;& V-(1+r)\cdot(\bar{p}_S+\tfrac{c_j}{2})+\tfrac{r}{n}\cdot\sum_{k\in S}(\bar{p}_S+\tfrac{c_k}{2}) \\
= \;& V-\left[1+r-\tfrac{s\cdot r}{n}\right]\cdot\bar{p}_S \\
+ \;& \tfrac{1}{2}\cdot\left[-(1+r)\cdot c_j+\tfrac{r}{n}\cdot\sum_{k\in S}c_k\right] \\
= \;& \tfrac{V}{2}+\tfrac{1}{2}\cdot\left[-(1+r)\cdot c_j+\tfrac{r}{n}\cdot\sum_{k\in S}c_k\right] \\
= \;& \tfrac{1}{2}\cdot\left[V-(1+r)\cdot c_j+\tfrac{r\cdot s}{n}\cdot\bar{c}_S\right]
\end{aligned}$$

Recall that $\bar{p}_j^S - c_j = \bar{p}_S - \frac{c_j}{2}$ for all $j \in S$. Through substitution in (5), the coalitional profit reduces as follows:

$$\begin{aligned}
v_\beta(S) &= \pi_S((\bar{p}_j^S)_{j\in S},\bar{p}_{N\setminus S}) \\
&= \tfrac{1}{2}\cdot\sum_{j\in S}(\bar{p}_S-\tfrac{c_j}{2})\cdot\left[V-(1+r)\cdot c_j+\tfrac{r\cdot s}{n}\cdot\bar{c}_S\right] \\
&= \frac{s\cdot\bar{p}_S}{2}\cdot\left[V+\tfrac{r\cdot s}{n}\cdot\bar{c}_S-(1+r)\cdot\bar{c}_S\right] \\
&- \left[V+\tfrac{r\cdot s}{n}\cdot\bar{c}_S\right]\cdot\tfrac{s\cdot\bar{c}_S}{4}+\tfrac{1+r}{4}\cdot\sum_{j\in S}(c_j)^2 \\
&= \tfrac{s\cdot V}{4}\cdot\tfrac{1}{[n+r\cdot(n-s)]}\cdot\Bigg[n\cdot V \\
&- \left[n+r\cdot(n-s)\right]\cdot\bar{c}_S\Bigg] \\
&- \left[n\cdot V+r\cdot s\cdot\bar{c}_S\right]\cdot\frac{s\cdot\bar{c}_S}{4\cdot n}+\tfrac{1+r}{4}\cdot\sum_{j\in S}(c_j)^2 \\
&= \tfrac{s}{4\cdot n}\cdot\tfrac{(n\cdot V)^2}{[n+r\cdot(n-s)]}-\tfrac{V}{2}\cdot\sum_{j\in S}c_j \\
&+ \tfrac{1+r}{4}\cdot\sum_{j\in S}(c_j)^2-\tfrac{r}{4\cdot n}\cdot\left[\sum_{j\in S}c_j\right]^2
\end{aligned}$$

This proves (14) concerning the coalitional worth $v_\beta(S)$ as well as the equivalent formula (15). □

In the context of (15), we remark, without going into details, that, concerning the first component, and sufficiently small r, the sum of individual worths approaches $\sum_{i\in N}(V-c_i)^2$ which always exceeds $n\cdot(V-\bar{c}_N)^2$ being the worth of the grand coalition. In other words, for r approaching to zero, the imputation set of the first component is empty. Nevertheless, the imputation set of the Bertrand Oligopoly Game is compensated by the imputation set of the second zero-normalized game being the Variance game.

Corollary 2.2: The α- and β-characteristic function forms $\langle N, v_\alpha\rangle$ and $\langle N, v_\beta\rangle$ coincide, that is $v_\alpha(S) = v_\beta(S)$ for all $S \subseteq N$.

Proof. Fix the coalition $S \subseteq N$, $S \neq \emptyset$. It remains to prove the inequality $v_\alpha(S) \geq v_\beta(S)$. We claim the following chain of (in)equalities:

$$\begin{aligned}
v_\alpha(S) &= \max_{p_S\in X_S}\min_{p_{N\setminus S}\in X_{N\setminus S}}\pi_S(p_S,p_{N\setminus S}) \\
&\geq \min_{p_{N\setminus S}\in X_{N\setminus S}}\pi_S((\bar{p}_j^S)_{j\in S},p_{N\setminus S}) \\
&= \pi_S((\bar{p}_j^S)_{j\in S},\bar{p}_{N\setminus S}) = v_\beta(S)
\end{aligned}$$

The last equality holds by (6) because of the construction of both vectors $(\bar{p}_j^S)_{j\in S}$ and $\bar{p}_{N\setminus S}$, whereas the last equality but one is due by (8) because $\bar{p}_{N\setminus S}$ is a minimizer of the minimization program $\min_{p_{N\setminus S}\in X_{N\setminus S}}\pi_S((\bar{p}_j^S)_{j\in S},p_{N\setminus S})$. □

III. THE CONVEXITY OF THE BERTRAND OLIGOPOLY TU-GAME

Let $\mathcal{P}(N) = \{S \mid S \subseteq N\}$ denote the *power set* of the player set N, consisting of all subsets of N. A cooperative TU game $\langle N, w\rangle$ is called *convex (supermodular)* if its characteristic function $w : \mathcal{P}(N) \to \mathbb{R}$ satisfies one of the following equivalent conditions ([7]): for all $S, T \subseteq N$,

$$w(S)+w(T)\leq w(S\cup T)+w(S\cap T)\quad\text{or}$$

for all $i \in N$ and all $S \subseteq T \subseteq N\setminus\{i\}$,

$$w(S\cup\{i\})-w(S)\leq w(T\cup\{i\})-w(T)\quad\text{or}$$

for all $i, j \in N$, $i \neq j$, and all $S \subseteq N\setminus\{i, j\}$,

$$w(S\cup\{i\})-w(S)\leq w(S\cup\{i,j\})-w(S\cup\{j\}) \tag{20}$$

First of all, the next example illustrates that Bertrand Oligopoly Games of form (11) needs not to be convex.

Example 3.1: We consider two three-firm Variance Games

(i) Consider three almost equal marginal costs $c_1 = c - 1$, $c_2 = c$ and $c_3 = c + 1$. The zero-normalized three-firm Variance Game is given by $v(\{1, 2\}) = v(\{2, 3\}) = \frac{1}{2}$ and $v(\{1, 3\}) = v(\{1, 2, 3\}) = 2$. This game is not convex since $v(N) - v(\{1, 3\}) = 0 < \frac{1}{2} = v(\{1, 2\}) - v(\{1\})$. The variance game is neither average convex since the constraint $\sum_{i\in S}[v(S) - v(S\setminus\{i\})] \leq \sum_{i\in S}[v(T) - v(T\setminus\{i\})]$ for all $S \subseteq T \subseteq N$ fails to hold whenever $S = \{1, 3\}$

and $T = N$.

(ii) Consider three very distinct marginal costs $c_1 = 1$, $c_2 = c+1$ and $c_3 = 2\cdot c+1$. The zero-normalized three-firm Variance Game is given by $v(\{1,2\}) = v(\{2,3\}) = \frac{c^2}{2}$, $v(\{1,3\}) = v(\{1,2,3\}) = 2\cdot c^2$. Notice that this variance game equals the former example multiplied by c^2.

A TU game $\langle N, w\rangle$ is said to be *non-essential (additive)* if its characteristic function $w : \mathcal{P}(N) \to \mathbb{R}$ satisfies $w(S) = \sum_{j\in S} w(\{j\})$ for all $S \subseteq N$, $S \neq \emptyset$. Obviously, non-essential games are convex since all convexity conditions are met as equalities. The main goal of this section is to prove the convexity (supermodularity) for Bertrand Oligopoly Games of the form (14).

For that purpose we assume that $n\cdot V > [n+r\cdot(n-1)]\cdot\max_{k\in N} c_k$ in order to guarantee that the constraint $n\cdot V > [n+r\cdot(n-s)]\cdot\bar{c}_S$ is met by every coalition $S \subseteq N$, $S \neq \emptyset$. According to the equivalent description (14), this type of Bertrand Oligopoly Game can be decomposed into four types of TU games as follows: for all $S \subseteq N$, $S \neq \emptyset$,

$$\begin{aligned} v_\beta(S) &= \tfrac{-V}{2}\cdot w_1(S) - \tfrac{r}{4\cdot n}\cdot w_2(S) \\ &+ \tfrac{1+r}{4}\cdot w_3(S) + \tfrac{V^2}{4\cdot(1+r)}\cdot w_4(S) \end{aligned} \tag{21}$$

Here both games $\langle N, w_k\rangle$, $k = 1,3$, are non-essential arising from the distinct marginal costs and their squares respectively, that is $w_1(S) = \sum_{j\in S} c_j$ and $w_3(S) = \sum_{j\in S}(c_j)^2$ for all $S \subseteq N$. In fact, both non-essential games are redundant for the convexity property. The game $\langle N, w_2\rangle$ is the square of the first non-essential game $\langle N, w_1\rangle$ in that $w_2(S) = (w_1(S))^2$ for all $S \subseteq N$. Generally speaking, the square of an non-essential game is convex too because the marginal contribution of a fixed player i with respect to variable coalitions $S \subseteq N\backslash\{i\}$ are non-decreasing with respect to set inclusion, that is

$$\begin{aligned} & w_2(S\cup\{i\}) - w_2(S) \\ =\; & \Big[w_1(S\cup\{i\})\Big]^2 - \Big[w_1(S)\Big]^2 \\ =\; & \Big[w_1(S) + w_1(\{i\})\Big]^2 - \Big[w_1(S)\Big]^2 \\ =\; & 2\cdot w_1(\{i\})\cdot w_1(S) + (w_1(\{i\}))^2 \quad \text{and so} \\ & \Big[w_2(S\cup\{i,j\}) - w_2(S\cup\{j\})\Big] \\ -\; & \Big[w_2(S\cup\{i\}) - w_2(S)\Big] \\ =\; & 2\cdot w_1(\{i\})\cdot w_1(\{j\}) \geq 0 \end{aligned} \tag{22}$$

for all $i,j\in N$, $i\neq j$, and all $S \subseteq N\backslash\{i,j\}$

Lemma 3.1: Given the substitutability parameter $r > 0$, define the n-person game $\langle N, w_4\rangle$ and the real-valued function $f : [0, \frac{1}{r_n}) \to \mathbb{R}$ by

$$w_4(S) = \tfrac{s\cdot n\cdot(1+r)}{n+r\cdot(n-s)} \qquad \text{for all } S \subseteq N, \tag{23}$$

$$f(x) = \frac{x}{1 - r_n\cdot x} \qquad \text{for all } x \in [0, \tfrac{1}{r_n}), \tag{24}$$

where $r_n = \frac{r}{n\cdot(1+r)}$

Then the following holds:

(i) $w_4(S) = f(s)$ for all $S \subseteq N$ with size s, $s = 0,1,2,\ldots,n$ and $f(n) = n\cdot(1+r)$

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and *strictly convex* on the interval $[0, \frac{1}{r_n})$ and consequently, the game $\langle N, w_4\rangle$ is *strictly convex* in that

$$f(s+2) - f(s+1) > f(s+1) - f(s) \tag{25}$$

for all $s = 0,1,2,\ldots,n-2$

(iii) The marginal returns of the function f satisfy

$$\begin{aligned} & f(s+1) - f(s) \\ =\; & \frac{n^2\cdot(1+r)^2}{[n+r\cdot(n-s)]\cdot[n+r\cdot(n-1-s)]} \end{aligned} \tag{26}$$

(iv) Applying (26) twice yields

$$\begin{aligned} & \Big[f(s+2) - f(s+1)\Big] - \Big[f(s+1) - f(s)\Big] \\ =\; & 2\cdot r\cdot n^2\cdot(1+r)^2\cdot[n+r\cdot(n-s)]^{-1} \\ \cdot\; & [n+r\cdot(n-1-s)]^{-1}\cdot[n+r\cdot(n-2-s)]^{-1} \end{aligned} \tag{27}$$

(v) $v_\beta(S) = \frac{1+r}{4}\cdot\frac{\left[\frac{V}{1+r}\cdot f(s) - s\cdot\bar{c}_S\right]^2}{f(s)} + \frac{1+r}{4}\cdot VAR_{\mathcal{C}}(S)$ provided $f(s) > \frac{(1+r)\cdot s}{V}\cdot\bar{c}_S$

Proof. Let $S \subseteq N$ be of size s, $s = 0,1,2,\ldots,n$. From (23)-(24), we derive

$$\begin{aligned} f(s) &= \frac{s}{1 - r_n\cdot s} = \frac{s}{1 - \frac{r\cdot s}{n\cdot(1+r)}} \\ &= \frac{s\cdot n\cdot(1+r)}{n\cdot(1+r) - r\cdot s} = \frac{s\cdot n\cdot(1+r)}{n+r\cdot(n-s)} = w_4(S) \end{aligned}$$

So,

$$\frac{V}{1+r}\cdot f(s)-s\cdot\bar{c}_S=\frac{s\cdot n\cdot V}{n+r\cdot(n-s)}-s\cdot\bar{c}_S$$
$$= s\cdot\left[\frac{n\cdot V}{n+r\cdot(n-s)}-\bar{c}_S\right]$$

Together with (15)-(16), this completes the proof of the coalitional worth $v_\beta(S)$. It is left to the reader to verify that the first and second derivative of the differentiable function $f(x)$ are given by $f'(x)=\frac{1}{(1-r_n\cdot x)^2}>0$ as well as $f''(x)=\frac{2\cdot r_n}{(1-r_n\cdot x)^3}>0$. Recall that a differentiable function is convex if and only if the second derivative is non-negative. □

In summary, so far, all four games $\langle N,w_k\rangle$, $k=1,2,3,4$, are convex (where the two non-essential games are redundant for the convexity property). Because the Bertrand Oligopoly Game of the form (21) is the difference of two convex games, it may fail to be convex itself. According to the proof of the next main theorem, convexity still holds for the Bertrand Oligopoly Game due to the existence of the underlying constraints.

Theorem 3.1: Suppose $n\cdot V>[n+r\cdot(n-1)]\cdot\max_{k\in N}c_k$. Then the Bertrand Oligopoly Game $\langle N,v_\beta\rangle$ of the form (21) is convex (supermodular).

Proof. In view of the decomposition (21), the Bertrand Oligopoly Game is convex if and only if the game $\langle N,n\cdot V^2\cdot w_4-r\cdot(1+r)\cdot w_2\rangle$ is convex. By Lemma 3.1(i), $w_4(S)=f(s)$ for all $S\subseteq N$, whereas (22) holds in the setting of the game $\langle N,w_2\rangle$ satisfying $w_2(\{k\})=c_k$ for all $k\in N$. In summary, the convexity property (20) applies to the Bertrand Oligopoly Game if and only if the following holds: for all $i,j\in N$, $i\neq j$, and all $s=0,1,2,\ldots,n-2$,

$$\left[f(s+2)-f(s+1)\right]-\left[f(s+1)-f(s)\right]$$
$$\geq\frac{r\cdot(1+r)}{n\cdot V^2}\cdot\left[2\cdot c_i\cdot c_j\right]\tag{28}$$

By assumption, $c_k\leq\frac{n\cdot V}{n+r\cdot(n-1)}$ for $k\in\{i,j\}$ and thus, it suffices to prove

$$\left[f(s+2)-f(s+1)\right]-\left[f(s+1)-f(s)\right]$$
$$\geq\frac{2\cdot n\cdot r\cdot(1+r)}{[n+r\cdot(n-1)]^2}\tag{29}$$

Recall the function $f(x)=\frac{x\cdot n\cdot(1+r)}{n+r\cdot(n-x)}$ and the results (26)–(27). Note that the expression at the right hand of (27) is non-decreasing in the variable coalition size s, attaining its minimum at $s=0$. It follows that

$$\left[f(s+2)-f(s+1)\right]-\left[f(s+1)-f(s)\right]$$
$$\geq\frac{2\cdot r\cdot n^2\cdot(1+r)^2}{[n+r\cdot n]\cdot[n+r\cdot(n-1)]\cdot[n+r\cdot(n-2)]}$$
$$=\frac{2\cdot r\cdot n\cdot(1+r)}{[n+r\cdot(n-1)]\cdot[n+r\cdot(n-2)]}$$
$$\geq\frac{2\cdot n\cdot r\cdot(1+r)}{[n+r\cdot(n-1)]^2}$$

This completes the proof of convexity for the Bertrand Oligopoly Game. □

With the specific choice (24) of the convex function f in mind, (29) is a sufficient condition for the convexity notion applied to the Bertrand Oligopoly Game of the form (21). Under the transformation $g(x)=r_n\cdot f(x)$, we have that (29) is fully equivalent to

$$\left[g(s+2)-g(s+1)\right]-\left[g(s+1)-g(s)\right]\geq 2\cdot(g(1))^2\tag{30}$$

Hence, in the framework of any not yet specified function $g:\mathbb{R}\to\mathbb{R}$, the convexity of g together with (30), are sufficient for the convexity of the Bertrand Oligopoly Game of the form $v_\beta(S)=\frac{-r}{4\cdot n}\cdot w_2(S)+\frac{V^2}{4}\cdot\frac{n}{r}\cdot g(s)$ for all $S\subseteq N$.

Proposition 3.1: The Variance game $\langle N,VAR_{\mathcal{C}}\rangle$ of (15) is not convex, unless all the firms have the same marginal costs yielding the zero game.

Proof. Suppose that at least two firms (but not all the firms) have the same marginal cost, say cost c. Let E denote the set of firms with the constant marginal cost c, that is $E=\{i\in N\mid c_i=c\}$. By assumption, $2\leq|E|<n$. Obviously, $\bar{c}_S=c$ and thus, $VAR_{\mathcal{C}}(S)=0$ for every $S\subseteq E$. For every $S\subseteq N$ satisfying $S\not\subseteq E$, it holds

$$\bar{c}(S)=\frac{1}{s}\cdot\sum_{k\in S}c_k=\frac{1}{s}\cdot\left[|S\cap E|\cdot c+\sum_{k\in S\setminus E}c_k\right]$$
$$=c+\frac{1}{s}\cdot\sum_{k\in S\setminus E}(c_k-c)\quad\text{and so,}$$

$$VAR_{\mathcal{C}}(S) = \sum_{k\in S\cap E}\Big[c - \bar{c}_S\Big]^2 + \sum_{k\in S\setminus E}\Big[c_k - \bar{c}_S\Big]^2$$

$$= \sum_{k\in S\cap E}(-\delta_S)^2 + \sum_{k\in S\setminus E}\Big[c_k - c - \delta_S\Big]^2$$

$$\text{(where} \quad \delta_S = \tfrac{1}{s}\cdot \sum_{k\in S\setminus E}(c_k - c))$$

$$= \sum_{k\in S\cap E}(\delta_S)^2 + \sum_{k\in S\setminus E}\Big[c_k - c\Big]^2$$

$$- \sum_{k\in S\setminus E} 2\cdot\delta_S\cdot\Big[c_k - c\Big] + \sum_{k\in S\setminus E}(\delta_S)^2$$

$$= s\cdot(\delta_S)^2 + \sum_{k\in S\setminus E}\Big[c_k - c\Big]^2 - 2\cdot s\cdot(\delta_S)^2$$

$$= \sum_{k\in S\setminus E}\Big[c_k - c\Big]^2 - s\cdot(\delta_S)^2$$

$$= \sum_{k\in S\setminus E}\Big[c_k - c\Big]^2 - \tfrac{1}{s}\cdot\Big[\sum_{k\in S\setminus E}(c_k - c)\Big]^2$$

For all $i \in E$, and all $S \subseteq N\backslash\{i\}$ satisfying $S \not\subseteq E$, it holds that $S \cup \{i\} \not\subseteq E$ as well as $(S\cup\{i\})\backslash E = S\backslash E$. From this, together with the latter equality, we derive on the one hand

$$\begin{aligned} & VAR_{\mathcal{C}}(S\cup\{i\}) - VAR_{\mathcal{C}}(S) \\ = \; & \Big[\frac{-1}{s+1} + \frac{1}{s}\Big]\cdot\Big[\sum_{k\in S\setminus E}(c_k - c)\Big]^2 \\ = \; & \tfrac{1}{s\cdot(s+1)}\cdot\Big[\sum_{k\in S\setminus E}(c_k - c)\Big]^2 \end{aligned}$$

Similarly, for all $i, j \in E$, and all $S \subseteq N\backslash\{i, j\}$ satisfying $S \not\subseteq E$, it holds that $S\cup\{i,j\} \not\subseteq E$ as well as $(S\cup\{i,j\})\backslash E = (S\cup\{j\})\backslash E = S\backslash E$ and so, we derive on the other

$$\begin{aligned} & VAR_{\mathcal{C}}(S\cup\{i,j\}) - VAR_{\mathcal{C}}(S\cup\{j\}) \\ = \; & \tfrac{1}{(s+1)\cdot(s+2)}\cdot\Big[\sum_{k\in S\setminus E}(c_k - c)\Big]^2 \end{aligned}$$

To complete the proof, choose $i, j \in E$. $i \neq j$, as well as any $\ell \in N\backslash E$. The former two equalities apply to $S = \{\ell\}$ since $S \not\subseteq E$. Then the strict inequalty $VAR_{\mathcal{C}}(S\cup\{i,j\}) - VAR_{\mathcal{C}}(S\cup\{j\}) < VAR_{\mathcal{C}}(S\cup\{i\}) - VAR_{\mathcal{C}}(S)$ holds due to $(s+1)\cdot(s+2) > s\cdot(s+1)$ as well as $\Big[\sum_{k\in S\setminus E}(c_k - c)\Big]^2 = \Big[c_\ell - c\Big]^2 > 0$. □

IV. The Shapley Value of the Bertrand Oligopoly TU-Game

The decomposition (21) of the Bertrand Oligopoly Game $\langle N, v_\beta\rangle$ into four types of games permits to determine its *Shapley value* $Sh(N, v_\beta)$ on basis of its linearity, efficiency, symmetry, and strategic equivalence. Generally speaking, the Shapley value $Sh(N, w) = (Sh_i(N, w))_{i\in N}$ of an arbitrary game $\langle N, w\rangle$ is given by an appropriate weighted, probabilistic sum of player's marginal contributions of the form $w(S\cup\{i\}) - w(S)$, $S \subseteq N\backslash\{i\}$, that is ([8]),for all $i \in N$

$$Sh_i(N, w) = \sum_{S\subseteq N\setminus\{i\}} p_n(s)\cdot\Big[w(S\cup\{i\}) - w(S)\Big] \tag{31}$$

where $p_n(s) = \frac{1}{n\cdot\binom{n-1}{s}}$ for all $s = 0, 1, 2, \ldots, n-1$. Due to its probabilistic interpretation, the Shapley value of any non-essential game $\langle N, w\rangle$ equals the individual's worth $w(\{i\})$, $i \in N$. Moreover, because of anonimity and efficiency, the Shapley value of the symmetric game $\langle N, w_4\rangle$ is fully determined by $Sh_i(N, w_4) = \frac{w_4(N)}{n} = 1 + r$ for all $i \in N$. Fourthly, the computation of the Shapley value of the game $\langle N, w_2\rangle$ proceeds by using (22) yielding for all $i \in N$

$$\begin{aligned} Sh_i(N, w_2) &= \sum_{S\subseteq N\setminus\{i\}} p_n(s)\cdot\Big[(c_i)^2 + 2\cdot c_i\cdot\sum_{j\in S} c_j\Big] \\ &= (c_i)^2 + 2\cdot c_i\cdot p_n(s)\cdot\sum_{S\subseteq N\setminus\{i\}}\sum_{j\in S} c_j \end{aligned}$$

Through the inverse order $\sum_{j\in N\setminus\{i\}} c_j \cdot \sum_{\substack{S\subseteq N\setminus\{i\}.\\ S\ni j}} p_n(s)$, together with two combinatorial steps, we arrive at $Sh_i(N, w_2) = c_i\cdot\sum_{j\in N} c_j = n\cdot c_i\cdot\bar{c}_N$. Finally, by applying the linearity of the Shapley value, we conclude the following:

Theorem 4.1: The Shapley value of the Bertrand Oligopoly Game of the form (21) is given by

$$Sh_i(N, v_\beta) = \tfrac{-V}{2}\cdot c_i - \tfrac{r}{4\cdot n}\cdot n\cdot c_i\cdot\bar{c}_N + \tfrac{1+r}{4}\cdot(c_i)^2 + \tfrac{V^2}{4} \tag{32}$$

In short, $Sh_i(N, v_\beta) = \frac{(V-c_i)^2}{4} + \frac{r}{4}\cdot c_i\cdot(c_i - \bar{c}_N)$ for all $i \in N$.

In words, the Shapley value of the Bertrand Oligopoly Game involves two types of payoffs to each firm i, $i \in N$, namely the square of the netto demand intercept $V - c_i$, as well as a proportional part c_i of the firm's deviation $c_i - \bar{c}_N$ from the average grand coalitional cost.

Theorem 4.2: The Shapley value of the Variance TU game $\langle N, VAR_{\mathcal{C}} \rangle$ of the form (15) is given by

$$Sh_i(N, VAR_{\mathcal{C}}) = a_n \cdot \bar{c}_n^{+i} - b_n \cdot \bar{c}_n^{-i} \text{where} \quad (33)$$

$$\bar{c}_n^{+i} = \sum_{j \in N \setminus \{i\}} \frac{(c_i - c_j)^2}{n-1} \quad (34)$$

$$\bar{c}_n^{-i} = \sum_{\substack{\{j,k\} \subseteq N \setminus \{i\}, \\ j \neq k}} \frac{(c_j - c_k)^2}{\binom{n-1}{2}} \quad (35)$$

$$a_n = \frac{n - H_n}{n} \quad \text{and} \quad b_n = \frac{\frac{n+1}{2} - H_n}{n} \quad (36)$$

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \quad (37)$$

Lemma 4.1: The game representation of the Variance TU game $\langle N, VAR_{\mathcal{C}} \rangle$ of the form (15) with respect to the basis of unanimity games $\{(N, u_T) \mid T \subseteq N, T \neq \emptyset\}$ is given by

$$VAR_{\mathcal{C}} = \sum_{\substack{T \subseteq N, \\ T \neq \emptyset}} \lambda_T^{VAR} \cdot u_T \quad (38)$$

where $\lambda_T^{VAR} = \frac{(-1)^t}{2} \cdot \sum_{\substack{\{j,k\} \subseteq T, \\ j \neq k}} \frac{(c_j - c_k)^2}{\binom{t}{2}}$ if $t \geq 2$, while $\lambda_{\{j\}}^{VAR} = 0$ for all $j \in N$.

Proof of Lemma 4.1 Through substitution of λ_T^{VAR}, $T \subseteq N$, $T \neq \emptyset$, we show that for all $S \subseteq N$, $S \neq \emptyset$,

$$VAR_{\mathcal{C}}(S) = \sum_{\substack{T \subseteq N, \\ T \neq \emptyset}} \lambda_T^{VAR} \cdot u_T(S) \quad (39)$$

or equivalently,$VAR_{\mathcal{C}}(S) = \sum_{\substack{T \subseteq S, \\ t \geq 2}} \lambda_T^{VAR}$.

Let $S \subseteq N$, $S \neq \emptyset$. For one-person coalitions $S = \{j\}$, $j \in N$, the latter equality holds because of $\lambda_{\{j\}}^{VAR} = 0 = VAR_{\mathcal{C}}(\{j\})$. In the remainder, let $s \geq 2$. On the one hand, we obtain the following chain of equations.

$$\sum_{\substack{T \subseteq S, \\ t \geq 2}} \lambda_T^{VAR} = \sum_{\substack{T \subseteq S, \\ t \geq 2}} \frac{(-1)^t}{t \cdot (t-1)} \cdot \sum_{\substack{\{j,k\} \subseteq T, \\ j \neq k}} (c_j - c_k)^2$$

$$= \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} (c_j - c_k)^2 \cdot \sum_{\substack{T \subseteq S, \\ \{j,k\} \subseteq T}} \frac{(-1)^t}{t \cdot (t-1)}$$

$$= \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} (c_j - c_k)^2 \cdot \sum_{t=2}^{s} \binom{s-2}{t-2} \cdot \frac{(-1)^t}{t \cdot (t-1)}$$

$$= \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} (c_j - c_k)^2 \cdot \sum_{t=2}^{s} \binom{s}{t} \cdot (-1)^t \cdot \frac{1}{s \cdot (s-1)}$$

$$= \tfrac{1}{s \cdot (s-1)} \cdot \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} (c_j - c_k)^2$$

$$\cdot \left[\sum_{t=0}^{s} \binom{s}{t} \cdot (-1)^t - \binom{s}{0} + \binom{s}{1} \right]$$

$$= \frac{1}{s \cdot (s-1)} \cdot \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} (c_j - c_k)^2$$

$$\cdot \left[((-1) + 1)^s - 1 + s \right]$$

$$= \tfrac{1}{s} \cdot \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} (c_j - c_k)^2$$

On the other hand,

$$VAR_{\mathcal{C}}(S) = \sum_{j \in S} \Big[c_j - \bar{c}(S) \Big]^2$$

$$= \sum_{j \in S} \Big[(c_j)^2 + (\bar{c}(S))^2 - 2 \cdot c_j \cdot \bar{c}(S) \Big]$$

$$= \sum_{j \in S} (c_j)^2 - s \cdot (\bar{c}(S))^2$$

$$= \sum_{j \in S} (c_j)^2 - \tfrac{1}{s} \cdot \Big[\sum_{j \in S} c_j \Big]^2$$

$$= \sum_{j \in S} (c_j)^2 - \tfrac{1}{s} \cdot \Big[\sum_{j \in S} (c_j)^2 + 2 \cdot \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} c_j \cdot c_k \Big]$$

$$= \tfrac{s-1}{s} \cdot \sum_{j \in S} (c_j)^2 - \tfrac{2}{s} \cdot \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} c_j \cdot c_k$$

$$= \tfrac{1}{s} \cdot \sum_{j \in S} (s-1) \cdot (c_j)^2 - \tfrac{2}{s} \cdot \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} c_j \cdot c_k$$

$$= \tfrac{1}{s} \cdot \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} \Big[(c_j)^2 + (c_k)^2 - 2 \cdot c_j \cdot c_k \Big]$$

$$= \tfrac{1}{s} \cdot \sum_{\substack{\{j,k\} \subseteq S, \\ j \neq k}} (c_j - c_k)^2$$

Both parts yield (39) for all $S \subseteq N$, $S \neq \emptyset$, and consequently, (38) holds. □

Proof of Theorem 4.2

Recall the well-known fact that the Shapley value of any unanimity game is determined by $Sh_i(N, u_T) = 0$ for all $i \in N \setminus T$, whereas $Sh_i(N, u_T) = \frac{1}{t}$ for all $i \in T$. Together with the linearity of the Shapley value and the game representation (38) of the Variance TU game, it follows that its Shapley value is given by, for all $i \in N$,

$$Sh_i(N, VAR_{\mathcal{C}}) = \sum_{\substack{T\subseteq N,\\ T\neq\emptyset}} \lambda_T^{VAR} \cdot Sh_i(N, u_T)$$

$$= \sum_{\substack{T\subseteq N,\\ i\in T}} \frac{\lambda_T^{VAR}}{t} = \sum_{\substack{T\subseteq N, t\geq 2,\\ i\in T}} \frac{(-1)^t}{2\cdot t} \cdot \sum_{\substack{\{j,k\}\subseteq T,\\ j\neq k}} \frac{(c_j-c_k)^2}{\binom{t}{2}}$$

$$= \sum_{\substack{T\subseteq N, t\geq 2,\\ i\in T}} \frac{(-1)^t}{t^2\cdot(t-1)} \cdot \sum_{\substack{\{j,k\}\subseteq T,\\ j\neq k}} (c_j-c_k)^2$$

$$= \sum_{\substack{\{j,k\}\subseteq N,\\ j\neq k}} (c_j-c_k)^2 \cdot \sum_{\substack{T\subseteq N, i\in T,\\ \{j,k\}\subseteq T}} \frac{(-1)^t}{t^2\cdot(t-1)}$$

Given any pair $\{j,k\} \subseteq N$, and player $i \in N$, we distinguish two cases. If $i \in \{j,k\}$, then

$$\sum_{\substack{T\subseteq N, i\in T,\\ \{j,k\}\subseteq T}} \frac{(-1)^t}{t^2\cdot(t-1)} = \sum_{\substack{T\subseteq N,\\ \{j,k\}\subseteq T}} \frac{(-1)^t}{t^2\cdot(t-1)}$$

$$= \sum_{t=2}^{n} \binom{n-2}{t-2} \cdot \frac{(-1)^t}{t^2\cdot(t-1)}$$

$$= \frac{1}{n\cdot(n-1)} \cdot \sum_{t=2}^{n} \binom{n}{t} \cdot \frac{(-1)^t}{t}$$

$$= \frac{1}{n-1} + \frac{1}{n\cdot(n-1)} \cdot \sum_{t=1}^{n} \binom{n}{t} \cdot \frac{(-1)^t}{t}$$

$$= \frac{1}{n-1} + \frac{1}{n\cdot(n-1)} \cdot \alpha_n = \frac{n+\alpha_n}{n\cdot(n-1)}$$

where $\alpha_n = \sum_{t=1}^{n} \binom{n}{t} \cdot \frac{(-1)^t}{t}$. If $i \notin \{j,k\}$, then

$$\sum_{\substack{T\subseteq N, i\in T,\\ \{j,k\}\subseteq T}} \frac{(-1)^t}{t^2\cdot(t-1)}$$

$$= \sum_{\substack{T\subseteq N,\\ \{i,j,k\}\subseteq T}} \frac{(-1)^t}{t^2\cdot(t-1)} = \sum_{t=3}^{n} \binom{n-3}{t-3} \cdot \frac{(-1)^t}{t^2\cdot(t-1)}$$

$$= \frac{1}{n\cdot(n-1)\cdot(n-2)} \cdot \sum_{t=3}^{n} \binom{n}{t} \cdot \frac{t-2}{t} \cdot (-1)^t$$

$$= \frac{-1}{(n-1)\cdot(n-2)}$$

$$+ \frac{1}{n\cdot(n-1)\cdot(n-2)} \cdot \sum_{t=1}^{n} \binom{n}{t} \cdot \frac{t-2}{t} \cdot (-1)^t$$

$$= \frac{-1}{(n-1)\cdot(n-2)} - \frac{1}{n\cdot(n-1)\cdot(n-2)}$$

$$+ \frac{1}{n\cdot(n-1)\cdot(n-2)} \cdot \left[-2 \cdot \sum_{t=1}^{n} \binom{n}{t} \cdot \frac{(-1)^t}{t}\right.$$

$$\left. + \sum_{t=0}^{n} \binom{n}{t} \cdot (-1)^t \right]$$

$$= \frac{-(n+1)}{n\cdot(n-1)\cdot(n-2)} + \frac{-2\cdot\alpha_n + ((-1)+1)^n}{n\cdot(n-1)\cdot(n-2)}$$

$$= \frac{-(n+1) - 2\cdot\alpha_n}{n\cdot(n-1)\cdot(n-2)}$$

We arrive at

$$Sh_i(N, VAR_{\mathcal{C}})$$

$$= \sum_{\substack{\{j,k\}\subseteq N, j\neq k,\\ i\in\{j,k\}}} (c_j-c_k)^2 \cdot \frac{n+\alpha_n}{n\cdot(n-1)}$$

$$- \sum_{\substack{\{j,k\}\subseteq N, j\neq k,\\ i\notin\{j,k\}}} (c_j-c_k)^2 \cdot \frac{(n+1)+2\cdot\alpha_n}{n\cdot(n-1)\cdot(n-2)}$$

It remains to pove that the sequence $\alpha_n = \sum_{t=1}^{n} \binom{n}{t} \frac{(-1)^t}{t}$ agrees with $\alpha_n = -\sum_{k=1}^{n} \frac{1}{k}$. It is left for the reader to check that the recursive relationship $\alpha_{n+1} = \alpha_n - \frac{1}{n+1}$ holds by using the combinatorial relationship $\binom{n+1}{t} = \binom{n}{t} + \binom{n}{t-1}$ for all $0 \leq t \leq n$. Together with $\alpha_1 = -1$, the claim follows immediately. This completes the full proof of formula (34) for the Shapley value of the Variance TU game. $\square$

V. TOTALLY BALANCEDNESS OF THE BERTRAND OLIGOPOLY TU-GAME

An arbitrary game $\langle N, w\rangle$ with characteristic function $w : \mathcal{P}(N) \to \mathbb{R}$ is said to be *balanced* if it holds $\sum_{S\in\mathcal{B}} \lambda_S \cdot w(S) \leq w(N)$ for every *balanced collection* $\mathcal{B} \subseteq \mathcal{P}(N)$ of coalitions and corresponding weights $(\lambda_S)_{S\in\mathcal{B}}$ satisfying $\lambda_S > 0$ for all $S \in \mathcal{B}$ and $\sum_{S\in\mathcal{B}, S\ni i} \lambda_S = 1$ for all $i \in N$.

Theorem 5.1: The Variance TU game $\langle N, VAR_{\mathcal{C}}\rangle$ of the form (15) is balanced (and hence, its core is non-empty)

Proof. Let $\mathcal{B}$ be a balanced collection of coalitions S, $S \in \mathcal{B}$, with corresponding weights $\lambda_S > 0$, $S \in \mathcal{B}$, satisfying $\sum_{S\in\mathcal{B}, S\ni i} \lambda_S = 1$ for all $i \in N$. By various appropriate computations we obtain the following chain of equalities:

$$
\begin{aligned}
&\sum_{S\in\mathcal{B}} \lambda_S \cdot VAR_{\mathcal{C}}(S) = \sum_{S\in\mathcal{B}} \lambda_S \cdot \sum_{i\in S} \Big[c_i - \bar{c}_S\Big]^2 \\
= &\sum_{i\in N} \sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \lambda_S \cdot \Big[c_i - \bar{c}_S\Big]^2 \\
= &\sum_{i\in N} \sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \lambda_S \cdot \Big[(c_i - \bar{c}_N) + (\bar{c}_N - \bar{c}_S)\Big]^2 \\
= &\sum_{i\in N} \sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \lambda_S \cdot \Big[(c_i - \bar{c}_N)^2 + (\bar{c}_N - \bar{c}_S)^2 \\
+ \;& 2\cdot (c_i - \bar{c}_N)\cdot(\bar{c}_N - \bar{c}_S)\Big] \\
= &\sum_{i\in N} \sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \lambda_S \cdot \Big[c_i - \bar{c}_N\Big]^2 + \sum_{i\in N} \sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \lambda_S \\
\cdot \;& \Big[(\bar{c}_N - \bar{c}_S)^2 + 2\cdot(c_i - \bar{c}_N)\cdot(\bar{c}_N - \bar{c}_S)\Big] \\
= &\sum_{i\in N} \Big[c_i - \bar{c}_N\Big]^2 \cdot \sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \lambda_S + \sum_{i\in N} \sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \lambda_S \\
\cdot \;& \Big[\bar{c}_N - \bar{c}_S\Big] \cdot \Big[(\bar{c}_N - \bar{c}_S) + 2\cdot(c_i - \bar{c}_N)\Big] \\
= &\sum_{i\in N} \Big[c_i - \bar{c}_N\Big]^2 + \sum_{S\in\mathcal{B}} \sum_{i\in S} \lambda_S \\
\cdot \;& \Big[\bar{c}_N - \bar{c}_S\Big] \cdot \Big[2\cdot c_i - \bar{c}_N - \bar{c}_S\Big] \\
= \;& VAR_{\mathcal{C}}(N) + \sum_{S\in\mathcal{B}} \lambda_S \cdot (\bar{c}_N - \bar{c}_S) \\
\cdot &\sum_{i\in S} \Big[2\cdot c_i - \bar{c}_S - \bar{c}_N\Big] \\
= \;& VAR_{\mathcal{C}}(N) + \sum_{S\in\mathcal{B}} \lambda_S \\
\cdot \;& (\bar{c}_N - \bar{c}_S) \cdot \Big[2\cdot s\cdot \bar{c}_S - s\cdot \bar{c}_S - s\cdot \bar{c}_N\Big] \\
= \;& VAR_{\mathcal{C}}(N) + \sum_{S\in\mathcal{B}} \lambda_S \\
\cdot \;& (\bar{c}_N - \bar{c}_S) \cdot \Big[s\cdot(\bar{c}_S - \bar{c}_N)\Big] \\
= \;& VAR_{\mathcal{C}}(N) - \sum_{S\in\mathcal{B}} \lambda_S \cdot s \cdot \Big[\bar{c}_N - \bar{c}_S\Big]^2 \\
\leq \;& VAR_{\mathcal{C}}(N)
\end{aligned}
$$

Because every subgame of the Bertrand Oligopoly Game is of the same type (associated with the same data restricted to the new player set), the Bertrand Oligopoly Game is called *totally balanced* (and so, every subgame has a non-emepty core).

VI. Concluding Remarks

The Bertrand Oligopoly situation with Shubik's demand functions has been modelled as a cooperative TU-game in [5], but only with reference to identical marginal costs for all firms. The current paper continues to study the general situation with distinct marginal costs. The complexity of the description of the associated cooperative game, as a result of solving two subsequent optimization problems, is compensated by decomposing the Bertrand Oligopoly Game into four types of games, namely two non-essential games, one symmetric game, and the square of one of these non-essential games. Although it concerns the difference of two convex games, it is shown that the Bertrand Oligopoly Game is convex too. Its current proof technique by decomposition differs from Lardon's proof of convexity for the symmetrical Bertrand Oligopoly Game. Surprisingly, under certain circumstances, the Bertrand Oligopoly Game agrees with the fundamental notion in statistics called Variance with respect to the distinct marginal costs. In the symmetric Bertrand Oligopoly Game this variance degenerates into the trivial zero game.

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